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Approximation of Analytic Functions by Bernstein-Type Operators*

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1. INTRODUCTION

Let $\{h_j(z)\}$ denote a sequence of complex-valued functions defined on $\bar{\Delta} = \{z : |z| \leqslant 1\}$. Define a matrix $(a_{nk}(z))$ for each $z \in \bar{\Delta}$ by the relations

$$a_{00}(z) = 1, \qquad a_{0k}(z) = 0, \qquad k > 0, \tag{1.1}$$

$$\prod_{j=1}^{n} (wh_j(z) + 1 - h_j(z)) = \sum_{k=0}^{n} a_{nk}(z) w^k.$$

The matrix (a_{nk}) is a generalization of the Lototsky matrix [1, 2]. The substitution $h_j = (1 + d_j)^{-1}$ gives the usual form when $\{h_j\}$ is a bounded sequence of complex constants.

The linear operator L_n associated with the transform (1.1) is defined, for each function f whose domain includes [0, 1], by

$$L_n(f; z) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) a_{nk}(z). \tag{1.2}$$

A recent paper of King [4] discussed conditions on a sequence of realvalued functions $\{h_j(x)\}$ which ensure the uniform convergence of $\{L_n(f; x)\}$ to

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$f(x)$, for each $f \in C[0, 1]$. King also pointed out that, when $h_j(x) = x$ ($j = 1, 2,\ldots$), L_n becomes the classical n-th order Bernstein polynomial [6]. Henceforth, we shall refer to (1.2) as the Lototsky–Bernstein operator.

The present paper concerns uniform approximation of analytic functions by means of Lototsky–Bernstein operators. In Section 2 we obtain very general conditions on $\{h_j(z)\}$ which ensure that $\{L_n(f; z)\}$ converges uniformly to $f(z)$ on the closed unit disk when $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $\sum_{k=0}^{\infty} | a_k | < \infty$. Also, uniform convergence of the operators to f, for f analytic in an elliptical region, is discussed.

In Section 3, similar results are given for a class of polynomial operators recently introduced by Stancu [7].

In the sequel, let $e_k(x) = x^k$, $k = 0, 1,\ldots$.

2. The Lototsky–Bernstein Operator

The central result of this section is the following;

Theorem 2.1. *Let $\{h_i(z)\}$ be a sequence of complex-valued functions having the following properties:*

$$h_i \text{ is analytic in } | z | < r, \qquad r > 1, \qquad i = 1, 2,\ldots; \tag{2.1}$$

$$h_i(1) = 1, \; i = 1, 2,\ldots; \tag{2.2}$$

$$h_i^{(v)}(0) \geqslant 0, \qquad v = 0, 1, 2,\ldots, \qquad i = 1, 2,\ldots; \tag{2.3}$$

$$\sum_{i=1}^{n} h_i'(1) = O(n) \tag{2.4}$$

and

the $(C,1)$ transform of $\{h_i(z)\}$ converges to z on a set of points having a limit point in the open unit disk. (2.5)

If L_n denotes the n-th Lototsky–Bernstein operator generated by $\{h_i(z)\}$ and if $f(z) = \sum_{k=0}^{\infty} a_k z^k$, with $\sum_{k=0}^{\infty} | a_k | < \infty$, then $\| L_n(f;) - f \| \to 0$ as $n \to \infty$, where $\|f\| = \max\{ | f(z) | : z \in \bar{\Delta}\}$.

Proof. A function f satisfying the hypotheses is of the form $f = f_1 - f_2 + if_3 - if_4$, where each f_j has positive Taylor coefficients. Therefore it suffices to prove the theorem in the case $a_k \geqslant 0$ for all k.

Write

$$P_n(x; z) = \prod_{i=1}^{n} (1 - h_i(x) + zh_i(x)).$$

Easy computations show that

$$L_n(e_0\,;x) = P_n(x;1) = 1;$$

$$L_n(e_1\,;x) = \frac{1}{n}\frac{\partial P_n(x;1)}{\partial z} = \frac{1}{n}\sum_{i=1}^{n} h_i(x);$$

$$L_n(e_2\,;x) = \frac{1}{n^2}\left(\frac{\partial^2 P_n(x;1)}{\partial z^2} + \frac{\partial P_n(x;1)}{\partial z}\right)$$

$$= \left(\frac{1}{n}\sum_{i=1}^{n} h_i(x)\right)^2 - \frac{1}{n^2}\sum_{i=1}^{n}(h_i(x))^2 + \frac{1}{n^2}\sum_{i=1}^{n} h_i(x).$$

In fact, for $k \geqslant 1$,

$$n^k L_n(e_k\,;x) = \sum_{m=0}^{n} m^k a_{nm}(x)$$

$$= \sum_{m=0}^{n}\sum_{t=1}^{k} \sigma_k{}^t\, m(m-1)\cdots(m-t+1)\, a_{nm}(x)$$

$$= \sum_{t=1}^{k} \sigma_k{}^t \frac{\partial^t P_n(x;1)}{\partial z^t}, \tag{2.6}$$

where $\sigma_k{}^t$ denotes a Stirling number of the second kind [3]. But $\sigma_k{}^t$ is a positive integer for $1 \leqslant t \leqslant k$ and $\sigma_k{}^1 = \sigma_k{}^k = 1$. Also (2.3) implies that

$$\frac{\partial^{v+s} P_n(0;1)}{\partial z^v \partial x^s} \geqslant 0, \qquad v = 1, 2,\ldots, \qquad s = 0, 1,\ldots, \qquad n = 1, 2,\ldots .$$

Therefore, $L_n^{(s)}(e_k\,;0) \geqslant 0$, $n = 1, 2,\ldots$, $k = 1, 2,\ldots$, $s = 0, 1,\ldots$. This fact with (2.1) and (2.6) yield the inequalities

$$|\,L_n(e_k\,;z)\,| \leqslant L_n(e_k\,;|\,z\,|) \leqslant L_n(e_k\,;1), \qquad \text{for} \quad |\,z\,| \leqslant 1,$$

$n = 1, 2,\ldots$, $k = 0, 1,\ldots$, Using the definition of $L_n(e_k\,;x)$ and (2.2) it is easy to see that $L_n(e_k\,;1) = 1$ for all n and k. Clearly, for $|\,z\,| \leqslant 1$ and $n = 1, 2,\ldots$,

$$L_n(f;z) = \sum_{k=0}^{\infty} a_k L_n(e_k\,;z)$$

and therefore the sequence $\{L_n(f;z)\}$ is uniformly bounded on $|\,z\,| \leqslant 1$. Now hypotheses (2.1)–(2.3) and (2.5) together with Vitali's theorem imply that the $(C, 1)$ transform of the sequence $\{h_i(z)\}$ is uniformly convergent to z on closed subsets of the open unit disk. In addition, since $0 \leqslant h_i(x) \leqslant 1$

for $0 \leqslant x \leqslant 1$ and $i = 1, 2, \ldots$, the operators are positive on [0, 1] (see [4]). It now follows that $L_n(f; x) \to f(x)$ for $0 \leqslant x \leqslant 1$ [4]. Therefore the functions $L_n(f; z)$ converge uniformly to $f(z)$ on each disk $|z| \leqslant p < 1$. Since the series

$$\sum_{k=0}^{\infty} a_k \sum_{v=0}^{\infty} \frac{L_n^{(v)}(e_k ; 0)}{v!} z^v$$

converges uniformly on $|z| \leqslant 1$, $|L_n'(f; z)| \leqslant L_n'(f; p)$ for $|z| \leqslant p \leqslant 1$. Next, for any $|z| \leqslant 1$, $p \leqslant |z| \leqslant 1$, $z = te^{i\alpha}$,

$$\begin{aligned} |L_n(f; z) - L_n(f; pe^{i\alpha})| &\leqslant \int_p^t |L_n'(f; xe^{i\alpha})| \, dx \\ &\leqslant L_n(f; t) - L_n(f; p) \\ &\leqslant (t - p) L_n'(f; 1). \end{aligned}$$

Thus the functions $L_n(f; z)$ will be equicontinuous in $|z| \leqslant 1$ if the sequence $\{L_n'(f; 1)\}$ is bounded. But (2.2) and easy computations show that

$$\begin{aligned} L_n'(f; 1) &= \sum_{k=0}^{n} f\left(\frac{k}{n}\right) a_{nk}'(1) \\ &= \left(f(1) - f\left(\frac{n-1}{n}\right)\right) \sum_{j=1}^{n} h_j'(1), \end{aligned}$$

and the boundedness of $\{L_n'(f; 1)\}$ follows from (2.4). Finally, since the $L_n(f; z)$ converge uniformly to $f(z)$ on each disk $|z| \leqslant p < 1$ and are continuous on $|z| \leqslant 1$, they converge uniformly on $|z| \leqslant 1$. This completes the proof.

LEMMA 2.2. *Let $h_j(z) = a_j z + b_j$ ($j = 1, 2, \ldots$), where a_j and b_j are complex constants. If g is a polynomial of degree k, then $L_n(g; z)$ is a polynomial of degree $\leqslant k$.*

Proof. Let

$$r_i(w, z) = h_i(w)(zh_i(w) + 1 - h_i(w))^{-1}$$

and it follows that

$$\frac{\partial P_n(w; z)}{\partial z} = P_n(w; z) \sum_{i=1}^{n} r_i(w, z). \tag{2.7}$$

Hence

$$\frac{\partial P_n(w; 1)}{\partial z} = ns_n(w),$$

where $s_n(w)$ denotes the $(C, 1)$ transform of the sequence $\{h_i(w)\}$.

After differentiating (2.7) j times with respect to z, we obtain

$$\begin{aligned}\frac{1}{n^{j+1}}\frac{\partial^{j+1}P_n(w;1)}{\partial z^{j+1}} &= \frac{1}{n^{j+1}}\sum_{v=0}^{j}\binom{j}{v}\frac{\partial^{j-v}P_n(w;1)}{\partial z^{j-v}}\sum_{i=1}^{n}\frac{\partial^v r_i(w,1)}{\partial z^v} \\ &= n^{-j}\frac{\partial^j P_n(w;1)}{\partial z^j}s_n(w) + R_n(w)\end{aligned} \tag{2.8}$$

with

$$R_n(w) = n^{-j-1}\sum_{v=1}^{j}\binom{j}{v}\frac{\partial^{j-v}P_n(w;1)}{\partial z^{j-v}}\sum_{i=1}^{n}\frac{\partial^v r_i(w;1)}{\partial z^v}.$$

Using (2.7) and (2.8) it is easy to see that $\partial^j P_n(w; 1)/\partial z^j$ is a polynomial in w of degree j. The conclusion follows from the linearity of L_n and (2.6) by induction.

We remark that if the sequence $\{h_j(w)\}$ does not consist only of linear factors, the operator $L_n(f; z)$ will not necessarily take polynomials of degree k into polynomials of degree $\leqslant k$.

With the aid of the above lemma, we can obtain, in a manner similar to that used for the Bernstein polynomials [6, p. 90], an analog of Kantorovitch's theorem.

THEOREM 2.3. *Let $\{L_n\}$ be the sequence of Lototsky–Bernstein operators generated by $\{h_j(w)\}$, where*

$$0 \leqslant h_j(x) \leqslant 1 \quad \textit{for} \quad 0 \leqslant x \leqslant 1, \qquad j = 1, 2,\ldots; \tag{2.9}$$

$$\frac{1}{n}\sum_{j=1}^{n} h_j(x) \to x \textit{ at two points of } [0, 1]; \textit{ and} \tag{2.10}$$

$$h_j(x) = a_j x + b_j, \qquad j = 1, 2,\ldots. \tag{2.11}$$

Let f be analytic on the interior of an ellipse with foci 0 *and* 1. *Then*

$$\lim_{n\to\infty} L_n(f; z) = f(z)$$

uniformly on any closed subset interior to the ellipse.

3. THE POLYNOMIAL OPERATOR $P_m^{(\alpha)}$

In a recent paper, Stancu [7] introduced a general class of positive, polynomial linear operators $P_m^{(\alpha)}$, where

$$P_m^{(\alpha)}(f; x) = \sum_{k=0}^{m} w_{m,k}(x; \alpha) f\left(\frac{k}{m}\right), \tag{3.1}$$

and

$$w_{m,k}(x;\alpha) = \binom{m}{k} \frac{\prod_{v=0}^{k-1}(x+v\alpha)\prod_{\beta=0}^{m-k-1}(1-x+\beta\alpha)}{(1+\alpha)(1+2\alpha)\cdots(1+[m-1]\alpha)}, \tag{3.2}$$

α being a parameter which may depend only on the natural number m. Clearly $P_m^{(\alpha)}(f;x)$ is a polynomial of degree m.

For $\alpha = -1/m$, (3.1) becomes the Lagrange interpolation polynomial corresponding to the function f and the equally spaced points k/m ($k = 0, 1,..., m$), while $\alpha = 0$ yields the classical Bernstein polynomial. It is also shown in [7] that the well-known Szasz–Mirakyan operator may be obtained as a limiting case of (3.1).

THEOREM 3.1. *Let* $0 \leqslant \alpha = \alpha(m) \to 0$ $(m \to \infty)$. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$ *with* $\sum_{k=0}^{\infty} | a_k | < \infty$. *Then* $\| P_m^{(\alpha)}(f;\,) - f \| \to 0$ *and, for* $| z | < 1$,

$$\left(\frac{m(1+\alpha)}{1+m\alpha}\right)(P_m^{(\alpha)}(f;z) - f(z)) = O(1)\ (m \to \infty). \tag{3.3}$$

Proof: As in the proof of Theorem 2.1, we may let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ with $a_k \geqslant 0$ for all k. Theorem 3.1 of [7] implies

$$D_v P_m^{(\alpha)}(e_k\,;0) \geqslant 0, \quad k = 0, 1,..., \quad v = 0, 1,..., \quad m = 1, 2,..., \tag{3.4}$$

where D_v denotes the operation of taking the v-th derivative. Next (3.4) and [7, p. 1182] yield

$$| P_m^{(\alpha)}(e_k\,;z) | \leqslant P_m^{(\alpha)}(e_k\,;|z|) \leqslant P_m^{(\alpha)}(e_k\,;1) = 1, \tag{3.5}$$

for $k = 0, 1,..., m = 1, 2,..., | z | \leqslant 1$. According to Theorem 4.1 of [7],

$$\lim_{m\to\infty} P_m^{(\alpha)}(f;x) = f(x), \qquad 0 \leqslant x \leqslant 1. \tag{3.6}$$

Using Theorem 3.1 of [7] and the assumption $a_k \geqslant 0$, $k = 0, 1,...,$ we obtain

$$\begin{aligned} | D_1 P_m^{(\alpha)}(f;1) | &= \sum_{j=1}^{m} \binom{m}{j} \sum_{v=0}^{j-1} (1+\alpha v)^{-1} \Delta_{1/m}^{j} f(0) \\ &\leqslant \sum_{j=1}^{m} \binom{m}{j} j \Delta_{1/m}^{j} f(0) \\ &= D_1 B_m(f;1) \to f'(1), \end{aligned}$$

where B_m is the m-th order Bernstein polynomial. Thus

$$\{D_1 P_m^{(\alpha)}(f;1)\} \text{ is bounded.} \tag{3.7}$$

The first part of Theorem 3.1 now follows from (3.4)–(3.7) just as in the proof of Theorem 2.1.

Let $0 < |z| = x < 1$. Then

$$\left|\frac{P_m^{(\alpha)}(f; z) - f(z)}{1 - z}\right| \leqslant \sum_{k=0}^{\infty} a_k \sum_{v=0}^{k} \frac{D_v P_m^{(\alpha)}(e_k\,; 0)}{v!} \left|\frac{z^v - z^k}{1 - z}\right|$$

$$\leqslant \sum_{k=0}^{\infty} a_k \sum_{v=0}^{k} \frac{D_v P_m^{(\alpha)}(e_k\,; 0)}{v!} \left(\frac{x^v - x^k}{1 - x}\right)$$

$$= \frac{P_m^{(\alpha)}(f; x) - f(x)}{1 - x},$$

where we have used Theorem 3.1 of [7] to assert that $P_m^{(\alpha)}(e_k\,; z)$ is a polynomial of degree $\leqslant k$. The above and Theorem 7.1 of [7] yield (3.3).

We note that Theorem 3.1 of [7] implies $P_m^{(\alpha)}$ maps polynomials of degree k into polynomials of degree $\leqslant k$ and this fact may be used to obtain the analog of Theorem 2.3 for $P_m^{(\alpha)}$.

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